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Prosiding Seminar Nasional Sains

Kajian Operator Komutator

Fatahillah
Universitas Indraprasta – PGRI
e-mail: drs.fatahillah@gmail.com

Info Artikel

Kata kunci:
Matriks, Invers, Komutator, Kartesian dan Bola

Abstrak

Objek dalam pandangan ini ada 4 hal, yaitu: 1. Momentum sudut, 2. Matriks Invers, 3. Komutator Kartesian dan 4. Komutator Bola. Tujuan utama adalah untuk memperlihatkan pembuktian tentang komutator jarang (tidak pernah) ditemukan pada buku-buku mekanika kuantum lainnya. Salah satu hasil nya adalah: $[\hat{x},\hat{p}_x]=[\hat{y},\hat{p}_y]=[\hat{z},\hat{p}_z]=i\hbar$, $[\hat{x},\hat{p}_y]=[\hat{y},\hat{p}_z]=[\hat{z},\hat{p}_x]=0$, $[\hat{x},\hat{p}]=[\hat{y},\hat{p}]=[\hat{z},\hat{p}]=i\hbar$, $[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z$, $[\hat{L}_y,\hat{L}_z]=i\hbar\hat{L}_x$, $[\hat{L}_z,\hat{L}_x]=i\hbar\hat{L}_y$, $\hat{L}\times\hat{L}=i\hbar\hat{L}$, $\vec{L}\times\vec{L}=\vec{0}$, $[\hat{L}^2,\hat{L}_x]=[\hat{L}^2,\hat{L}_y]=[\hat{L}^2,\hat{L}_z]=0$ dan $[\hat{L}^2,\hat{L}]=0$, $\hat{L}_x=-i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_y=-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_z=-i\hbar\frac{\partial}{\partial\varphi}$, $\hat{L}^2=-\hbar^2\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$.

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PENDAHULUAN

Pada tulisan ini akan dibahasan dari 2 macam bentuk operator komutator yaitu operator komutator kartesian dan operator komutator bola, penjelasannya dalam bentuk soal pembuktian operator komutator.

Penulisan ini juga didasarkan hasil konsultasi penulis dengan Prof Emerittus. Pantur Silaban, Ph.D ditahun 2001 yang bertempat fakultas teknik UKRIDA.

Disini akan dibuktikan bahwa:

$[\hat{x},\hat{p}_x]=[\hat{y},\hat{p}_y]=[\hat{z},\hat{p}_z]=i\hbar$, $[\hat{x},\hat{p}_y]=[\hat{y},\hat{p}_z]=[\hat{z},\hat{p}_x]=0$, $[\hat{x},\hat{p}]=[\hat{y},\hat{p}]=[\hat{z},\hat{p}]=i\hbar$, $[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z$, $[\hat{L}_y,\hat{L}_z]=i\hbar\hat{L}_x$, $[\hat{L}_z,\hat{L}_x]=i\hbar\hat{L}_y$, $\hat{L}\times\hat{L}=i\hbar\hat{L}$, $\vec{L}\times\vec{L}=\vec{0}$, $[\hat{L}^2,\hat{L}_x]=[\hat{L}^2,\hat{L}_y]=[\hat{L}^2,\hat{L}_z]=0$ dan $[\hat{L}^2,\hat{L}]=0$, $\hat{L}_x=-i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_y=-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_z=-i\hbar\frac{\partial}{\partial\varphi}$, $\hat{L}^2=-\hbar^2\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$

METODE PENELITIAN

Metoda pembahasan ini berdasarkan studi literature dan pengembangannya yang lebih mendalam berdasarkan teori-teori yang pernah didapat dalam bentuk menjawab soal pembuktian.

HASIL PENELITIAN

Operator Comutator

Operator ini banyak diaplikasikan dalam momentum sudut dalam mekanika kuantum, bukan fisika kuantum dimana *"mekanika kuantum"* lebih ditekankan kepada pembahasan matematisnya seperti salah satunya adalah *"operator komutator"*.

Momentum sudut

Menurut mekanika newton ada kemiripan antara momentum linier dan momentum sudut seperti: Gaya: $\vec{F} = \frac{d\vec{p}}{dt}$, dan Momen Gaya: $\vec{\tau} = \frac{d\vec{L}}{dt}$, sehingga ada kemiripan diantaranya seperti: $\left.\begin{matrix}\vec{F} \approx \vec{\tau} \\ \vec{p} \approx \vec{L}\end{matrix}\right\}$ dimana: $\vec{p}$ = momentum linier dan $\vec{L}$ = momentum sudut dengan hubungan sbb:

$\vec{L} = \vec{r} \times \vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}$, dan komponen vektor momentum sudutnya:

$\left.\begin{matrix} L_x = yp_z - zp_y \\ L_y = zp_x - xp_z \\ L_z = xp_y - yp_x \end{matrix}\right\}$ dan: $L^2 = L_x^2 + L_y^2 + L_z^2$, dan sebagai operator berbentuk:

$\hat{L} = \begin{bmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{bmatrix}$ sehingga: $\hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \left(yp_z - zp_y\right)^2 + (zp_x - xp_z)^2 + \left(xp_y - yp_x\right)$

Operator momentum

Pada penurunan persamaan Schrodinger didapatkan hasil bahwa operator momentum linier adalah: $\hat{p} = -i\hbar\vec{\nabla} = -i\hbar \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$, dan operator momentum sudut: $\hat{L} = -i\hbar \begin{bmatrix} y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{bmatrix}$

$= \begin{bmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{bmatrix}$ dimana: $\begin{bmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{bmatrix} = -i\hbar \begin{bmatrix} y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{bmatrix} = -i\hbar\left(\hat{r} \times \hat{\nabla}\right)$

Definisi dan Lambang Operator Komutator

Komutator bekerja terhadap operator-operator yang dilambangkan dan didefinisikan sbb:
$[A, B] = AB - BA$, dimana: A dan B adalah operator.
Berdasarkan sistim koordinatnya maka ada 2 macam operator komutator yaitu Operator Komutator Kartesian dan Operator Komutator Bola
Berdasarkan sifatnya maka ada 2 macam komutator, yaitu:

1. Operator komutator kompatibel bila $[A, B] = 0$ (compatible commutator) dan
2. Operator komutator non kompatibel, bila $[A, B] \neq 0$, dimana: $A \neq B$ (noncompatible commutator).

Marilah kita membahas operator ini sekalian membahas contoh-contoh soalnya.

Cartesian Comutator (Komutator Kartesian)

Disini soal dibagi dalam 2 bagian yaitu pembahasan soal fundamental yang hasilnya akan akan diterapkan sebagai dasar dalam pembuktian hukum-hukum Cartesian Comutator dan spherical Comutator.

Fundamental Komutator Kartesian:

Buktikan bahwa: a). $[a, xy] = x[a, y] + [a, x]y$, b). $[ax, y] = a[x, y] + [a, y]x$, c). $[ax + xb, x] = [ax, x] + [x, b]x$ dan d). $[a + b, c + d] = [a, c] + [a, d] + [b, c] + [b, d]$

Bukti:

a) $[a, xy]$ = $axy - xya$ = $axy - xay$ + $xay - xya$ = $(ax - xa)y$ + $x(ay - ya)$
= $x(ay - ya)$ + $(ax - xa)y$ = $x[a, y]$ + $[a, x]y$
Jadi: $[a, xy] = x[a, y] + [a, x]y$

b) $[ax, y]$ = $axy - yax$ = $axy - ayx$ + $ayx - yax$ = $(axy - ayx)$ + $(ayx - yax)$
= $a(xy - yx)$ + $(ay - ya)x$ = $a[x, y]$ + $[a, y]x$
Jadi: $[ax, y]$ = $a[x, y]$ + $[a, y]x$

c) $[ax + xb, x]$ = $(ax + xb)x - x(ax + xb)$ = axx + $xbx - xax - xxb$
= $axx - xax$ + $xbx - xxb$ = $(axx - xax)$ + $(xbx - xxb)$ = $[ax, x]$ + $[xb, x]$
Jadi: $[ax + xb, x]$ = $[ax, x]$ + $[xb, x]$

d) $[A + B, C + D]$ = $(A + B)(C + D) - (C + D)(A + B)$
= AC + AD + BC + $BD - CA - CB - DA - DB$
= $AC - CA$ + $AD - DA$ + $BC - CB$ + $BD - DB$
= $(AC - CA)$ + $(AD - DA)$ + $(BC - CB)$ + $(BD - DB)$
= $[A, C]$ + $[A, D]$ + $[B, C]$ + $[B, D]$
Jadi: $[A + B, C + D]$ = $[A, C]$ + $[A, D]$ + $[B, C]$ + $[B, D]$

e) $[AB, CD]$ = $C[AB, D]$ + $[AB, C]D$ = $C([AB, D])$ + $([AB, C])D$
= $C(A[B, D] + [A, D]B)$ + $(A[B, C] + [A, C]B)D$
= $CA[B, D]$ + $C[A, D]B$ + $A[B, C]D$ + $[A, C]BD$
Jadi: $[AB, CD]$ = $CA[B, D]$ + $C[A, D]B$ + $A[B, C]D$ + $[A, C]BD$

Aplikasi Fundamental Cartesian Comutator

1. Buktikanlah bahwa: $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}, \frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial^2 \psi}{\partial z \partial x}$ dan $\frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial^2 \psi}{\partial z \partial y}$ bila:
$\psi = e^{i\{k(x+y+z)-\omega t\}} = \psi(x, y, z, t)$
Bukti:
$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left(e^{i\{k(x+y+z)-\omega t\}}\right)\right\} = ik\frac{\partial}{\partial x}\left(e^{i\{k(x+y+z)-\omega t\}}\right)$
$= -k^2 e^{i\{k(x+y+z)-\omega t\}} = -k^2 \psi$
$= -k^2 \psi$ dan $\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\left(e^{i\{k(x+y+z)-\omega t\}}\right)\right\} = ik\frac{\partial}{\partial y}\left(e^{i\{k(x+y+z)-\omega t\}}\right)$
$= -k^2 e^{i\{k(x+y+z)-\omega t\}} = -k^2 \psi$
Jadi: $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$ → terbukti.
Dengan cara yang sama akan terbukti bahwa: $\frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial^2 \psi}{\partial z \partial x}$ dan $\frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial^2 \psi}{\partial z \partial y}$

2. Buktikanlah bahwa: a). $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$
Bukti:
$[\hat{x}, \hat{p}_x]\psi = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi = \left\{x\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)x\right\}\psi$
$= -i\hbar\left\{x\left(\frac{\partial \psi}{\partial x}\right) - \left(\frac{\partial}{\partial x}\right)(x\psi)\right\} = -i\hbar x\left(\frac{\partial \psi}{\partial x}\right) + i\hbar\left(\frac{\partial}{\partial x}\right)(x\psi) = -i\hbar x\frac{\partial \psi}{\partial x}$
$+ i\hbar\left(\frac{\partial x}{\partial x}\psi + x\frac{\partial \psi}{\partial x}\right) = -i\hbar x\frac{\partial \psi}{\partial x} + i\hbar\frac{\partial x}{\partial x}\psi + i\hbar x\frac{\partial \psi}{\partial x} = i\hbar\frac{\partial x}{\partial x}\psi = i\hbar\psi$
Jadi: $[\hat{x}, \hat{p}_x] = i\hbar$ → terbukti.
Dengan cara yang sama akan terbukti bahwa: $[\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$

3. Buktikanlah bahwa: $[\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_x] = 0$
Bukti:
$[\hat{x}, \hat{p}_y]\psi = [x, \hat{p}_y]\psi = (x\hat{p}_y - \hat{p}_y x)\psi = \left\{x\left(-i\hbar\frac{\partial}{\partial y}\right) - \left(-i\hbar\frac{\partial}{\partial y}\right)x\right\}\psi$
$= -i\hbar\left\{x\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}(x\psi)\right\} = -i\hbar x\frac{\partial \psi}{\partial y} + i\hbar\frac{\partial}{\partial y}(x\psi) = -i\hbar x\frac{\partial \psi}{\partial y} + i\hbar\frac{\partial x}{\partial y}\psi + i\hbar x\frac{\partial \psi}{\partial y}$
$= i\hbar\frac{\partial x}{\partial y}\psi = i\hbar(0)\psi = 0$, dengan cara yang sama akan terbukti bahwa: $[\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_x] = 0$
Jadi: $[\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_x] = 0$ → terbukti.

3. Buktikanlah bahwa: $[\hat{x}, \hat{p}] = [\hat{y}, \hat{p}] = [\hat{z}, \hat{p}] = i\hbar$
Bukti:

$[\hat{x},\hat{p}]\psi = (x\hat{p}-\hat{p}x)\psi = \left(x\begin{bmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{bmatrix}-\begin{bmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{bmatrix}x\right)\psi = \left(\begin{bmatrix}x\hat{p}_x\\ x\hat{p}_y\\ x\hat{p}_z\end{bmatrix}-\begin{bmatrix}\hat{p}_x x\\ \hat{p}_y x\\ \hat{p}_z x\end{bmatrix}\right)\psi$

$= \begin{bmatrix}x\hat{p}_x-\hat{p}_x x\\ x\hat{p}_y-\hat{p}_y x\\ x\hat{p}_z-\hat{p}_z x\end{bmatrix}\psi = \begin{bmatrix}[x,\hat{p}_x]\\ [x,\hat{p}_y]\\ [x,\hat{p}_z]\end{bmatrix}\psi = \begin{bmatrix}[x,\hat{p}_x]\psi\\ [x,\hat{p}_y]\psi\\ [x,\hat{p}_z]\psi\end{bmatrix} = \begin{bmatrix}i\hbar\psi\\ 0\psi\\ 0\psi\end{bmatrix} = \begin{bmatrix}i\hbar\psi\\ 0\\ 0\end{bmatrix} = i\hbar\psi$, atau: $[\hat{x},\hat{p}]\psi$
$= i\hbar\psi$

Jadi: $[\hat{x},\hat{p}] = i\hbar \rightarrow$ terbukti.

Dengan cara yang sama dapat dihasilkan bahwa: $[\hat{x},\hat{p}] = [\hat{y},\hat{p}] = [\hat{z},\hat{p}] = i\hbar$

4. Buktikanlah bahwa: $[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x$ dan $[\hat{L}_z,\hat{L}_x] = i\hbar\hat{L}_y$

Bukti:

$[\hat{L}_x,\hat{L}_y]\psi = [\hat{y}\hat{p}_z-\hat{z}\hat{p}_y, \hat{z}\hat{p}_x-\hat{x}\hat{p}_z]\psi = [\hat{y}\hat{p}_z,\hat{z}\hat{p}_x]\psi + [\hat{z}\hat{p}_y,\hat{x}\hat{p}_z]\psi - [\hat{z}\hat{p}_y,\hat{z}\hat{p}_x]\psi$
$-[\hat{y}\hat{p}_z,\hat{x}\hat{p}_z]\psi = I_1 + I_2 - I_3 - I_4$

Dimana:

$I_1 = [\hat{y}\hat{p}_z,\hat{z}\hat{p}_x]\psi = [y\hat{p}_z, z\hat{p}_x]\psi = (y\hat{p}_z z\hat{p}_x - z\hat{p}_x y\hat{p}_z)\psi = y\hat{p}_z z\hat{p}_x\psi - z\hat{p}_x y\hat{p}_z\psi$
$= y\left(-i\hbar\frac{\partial}{\partial z}\right)z\left(-i\hbar\frac{\partial}{\partial x}\right)\psi - z\left(-i\hbar\frac{\partial}{\partial x}\right)y\left(-i\hbar\frac{\partial}{\partial z}\right)\psi = -\hbar^2 y\frac{\partial}{\partial z}\left(z\frac{\partial\psi}{\partial x}\right)$
$+ \hbar^2 z\frac{\partial}{\partial x}\left(y\frac{\partial\psi}{\partial z}\right) = -\hbar^2 y\frac{\partial z}{\partial z}\frac{\partial\psi}{\partial x} - \hbar^2 yz\frac{\partial^2\psi}{\partial z\partial x} + \hbar^2 z\frac{\partial y}{\partial x}\frac{\partial\psi}{\partial z} + \hbar^2 zy\frac{\partial^2\psi}{\partial x\partial z} = -\hbar^2 y\frac{\partial z}{\partial z}\frac{\partial\psi}{\partial x}$
$+ \hbar^2 z\frac{\partial y}{\partial x}\frac{\partial\psi}{\partial z} - \hbar^2 yz\frac{\partial^2\psi}{\partial x\partial z} + \hbar^2 yz\frac{\partial^2\psi}{\partial x\partial z} = -\hbar^2 y\frac{\partial\psi}{\partial x} + \hbar^2 z(0)\frac{\partial\psi}{\partial z} = -\hbar^2 y\frac{\partial\psi}{\partial x}$

Dengan cara yang sama dapat dihasilkan bahwa:

$I_2 = \hbar^2 x\frac{\partial\psi}{\partial y}$, $I_3 = [\hat{z}\hat{p}_y,\hat{z}\hat{p}_x]\psi = [\hat{y}\hat{p}_z,\hat{x}\hat{p}_z]\psi = I_4 = 0$

Sehingga:

$[\hat{L}_x,\hat{L}_y]\psi = I_1 + I_2 - I_3 - I_4 = -\hbar^2 y\frac{\partial\psi}{\partial x} + \hbar^2 x\frac{\partial\psi}{\partial y} + 0 + 0 = \hbar^2 x\frac{\partial\psi}{\partial y} - \hbar^2 y\frac{\partial\psi}{\partial x}$
$= \hbar(-ih)x\frac{\partial\psi}{\partial y}$
$- i\hbar(-ih)y\frac{\partial\psi}{\partial x} = i\hbar\left\{(-ih)x\frac{\partial\psi}{\partial y} - (-ih)y\frac{\partial\psi}{\partial x}\right\} = i\hbar\left\{x\left(-ih\frac{\partial\psi}{\partial y}\right) - y\left(-ih\frac{\partial\psi}{\partial x}\right)\right\}$
$= i\hbar\left\{x\left(-ih\frac{\partial}{\partial y}\right) - y\left(-ih\frac{\partial}{\partial x}\right)\right\}\psi = i\hbar\{x\hat{p}_y - y\hat{p}_x\}\psi = i\hbar\hat{L}_z\psi$, atau: $[\hat{L}_x,\hat{L}_y]\psi$
$= i\hbar\hat{L}_z\psi$

Jadi: $[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z$

Maka dengan cara yang sama dapat dihasilkan: $[\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x$ dan $[\hat{L}_z,\hat{L}_x] = i\hbar\hat{L}_y$

5. Buktikanlah bahwa: a). $\hat{L}\times\hat{L} = i\hbar\hat{L}$ dan b). $\vec{L}\times\vec{L} = \vec{0}$

Bukti:

a) $\hat{L}\times\hat{L} = \begin{bmatrix}\hat{L}_x\\ \hat{L}_y\\ \hat{L}_z\end{bmatrix}\times\begin{bmatrix}\hat{L}_x\\ \hat{L}_y\\ \hat{L}_z\end{bmatrix} = \begin{bmatrix}\hat{L}_y\hat{L}_z-\hat{L}_z\hat{L}_y\\ \hat{L}_z\hat{L}_x-\hat{L}_x\hat{L}_z\\ \hat{L}_x\hat{L}_y-\hat{L}_y\hat{L}_x\end{bmatrix} = \begin{bmatrix}[\hat{L}_y,\hat{L}_z]\\ [\hat{L}_z,\hat{L}_x]\\ [\hat{L}_x,\hat{L}_y]\end{bmatrix} = \begin{bmatrix}i\hbar\hat{L}_x\\ i\hbar\hat{L}_y\\ i\hbar\hat{L}_z\end{bmatrix} = i\hbar\begin{bmatrix}\hat{L}_x\\ \hat{L}_y\\ \hat{L}_z\end{bmatrix} = i\hbar\hat{L}$

Jadi: $\hat{L}\times\hat{L} = i\hbar\hat{L}$ (terbukti)

Sebaliknya dalam bentuk vektor (bukan operator vektor lagi), maka:

b) $\vec{L}\times\vec{L} = \begin{bmatrix}L_x\\ L_y\\ L_z\end{bmatrix}\times\begin{bmatrix}L_x\\ L_y\\ L_z\end{bmatrix} = \begin{bmatrix}L_yL_Z-L_yL_Z\\ L_zL_x-L_zL_x\\ L_xL_y-L_xL_y\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} = \vec{0}$

Jadi: $\vec{L}\times\vec{L} = \vec{0}$

9. Buktikanlah bahwa: $[\hat{L}^2,\hat{L}_x] = [\hat{L}^2,\hat{L}_y] = [\hat{L}^2,\hat{L}_z] = 0$

Bukti:

$[\hat{L}^2,\hat{L}_x] = [\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2,\hat{L}_x] = [\hat{L}_x^2,\hat{L}_x] + [\hat{L}_y^2,\hat{L}_x] + [\hat{L}_z^2,\hat{L}_x]$
$= (\hat{L}_x[\hat{L}_x,\hat{L}_x] + [\hat{L}_x,\hat{L}_x]\hat{L}_x) + (\hat{L}_y[\hat{L}_y,\hat{L}_x] + [\hat{L}_y,\hat{L}_x]\hat{L}_y) + (\hat{L}_z[\hat{L}_z,\hat{L}_x] + [\hat{L}_z,\hat{L}_x]\hat{L}_z)$
$= \{\hat{L}_x(0) + (0)\hat{L}_x\} + (\hat{L}_y[\hat{L}_y,\hat{L}_x] - [\hat{L}_x,\hat{L}_y]\hat{L}_y) + (\hat{L}_z[\hat{L}_z,\hat{L}_x] - [\hat{L}_x,\hat{L}_z]\hat{L}_z)$
$= 0 + \{-i\hbar\hat{L}_y\hat{L}_z - i\hbar\hat{L}_z\hat{L}_y\} + \{i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_y\hat{L}_z\} = -i\hbar\hat{L}_y\hat{L}_z - i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_z\hat{L}_y$

+ $i\hbar\hat{L}_y\hat{L}_z$ = – $i\hbar\hat{L}_y\hat{L}_z$ + $i\hbar\hat{L}_y\hat{L}_z$ – $i\hbar\hat{L}_z\hat{L}_y$ + $i\hbar\hat{L}_z\hat{L}_y$ = 0
Sehingga: $[\hat{L}^2, \hat{L}_x]$ = 0, dengan cara yang sama, maka: $[\hat{L}^2, \hat{L}_y]$ = $[\hat{L}^2, \hat{L}_z]$ = 0
Jadi: $[\hat{L}^2, \hat{L}_x]$ = $[\hat{L}^2, \hat{L}_y]$ = $[\hat{L}^2, \hat{L}_z]$ = 0 → terbukti.

10. Buktikanlah bahwa: $[\hat{L}^2, \hat{L}]$ = 0
    Bukti:
    $[\hat{L}^2, \hat{L}] = \hat{L}[\hat{L}, \hat{L}] + [\hat{L}, \hat{L}]\hat{L} = \hat{L}(0) + (0)\hat{L} = 0$
    Jadi: $[\hat{L}^2, \hat{L}]$ = 0 → terbukti.

Sperical Comutator Operator

Sebelum membahas Sperical Comutator Operator ada baiknya bicara koordinat bola terlebih dahulu.

Koordinat Bola

Hubungan komponen koordinat kartesian kekomponen koordinat bola dinyatakan sebagai berikut:

$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi = x(r,\theta,\varphi) \\ y &= r\sin\theta\sin\varphi = y(r,\theta,\varphi) \\ z &= r\cos\theta = z(r,\theta) \end{aligned}\right\},\quad \text{maka:}\quad \begin{aligned} \frac{\partial x}{\partial r} &= \sin\theta\cos\varphi \\ \frac{\partial y}{\partial r} &= \sin\theta\sin\varphi \\ \frac{\partial z}{\partial r} &= \cos\theta \end{aligned} \quad\rightarrow\quad \begin{aligned} \frac{\partial x}{\partial \theta} &= r\cos\theta\cos\varphi \\ \frac{\partial y}{\partial \theta} &= r\cos\theta\sin\varphi \\ \frac{\partial z}{\partial \theta} &= -r\sin\theta \end{aligned} \quad\rightarrow$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$$

$$\frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi$$

$$\frac{\partial z}{\partial \varphi} = 0$$

Atau:

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \varphi}d\varphi \\ dz &= \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \varphi}d\varphi \end{aligned}\right\}, \text{ dan matriksnya: } \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} \begin{bmatrix} dr \\ d\theta \\ d\varphi \end{bmatrix} \ldots\ldots\ldots.. \quad (1)$$

Persamaan inversnya adalah:

$$\left.\begin{aligned} r &= r(x,y,z) \\ \theta &= \theta(x,y,z) \\ \varphi &= \varphi(x,y,z) \end{aligned}\right\} \text{ atau: } \begin{aligned} dr &= \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz \\ d\theta &= \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy + \frac{\partial \theta}{\partial z}dz \\ d\varphi &= \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz \end{aligned} \rightarrow \begin{bmatrix} dr \\ d\theta \\ d\varphi \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \ldots\ldots \quad (2)$$

Jadi:

$$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix}^{-1} \text{ atau: } \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{bmatrix}^{-1}$$

Bila:

$$D = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

= $\sin\theta\cos\varphi\, r^2\sin^2\theta\cos\varphi$ + $r\cos\theta\cos\varphi\,(r\sin\theta\cos\theta\cos\varphi)$ – $r\sin\theta\sin\varphi\,(-r\sin\varphi)$
= $r^2\sin\theta\sin^2\theta\cos\varphi\cos\varphi$ + $r^2\sin\theta\cos\theta\cos\theta\cos\varphi\cos\varphi$ + $r^2\sin\theta\sin\varphi\sin\varphi$
= $r^2\sin\theta\sin^2\theta\cos^2\varphi$ + $r^2\sin\theta\cos^2\theta\cos^2\varphi$ + $r^2\sin\theta\sin^2\varphi$
= $r^2\sin\theta\,\{(\sin^2\theta + \cos^2\theta)\cos^2\varphi + \sin^2\varphi\}$ = $r^2\sin\theta\,\{\cos^2\varphi + \sin^2\varphi\}$ = $r^2\sin\theta$, atau: $D = r^2\sin\theta$, sehingga:

Maka:
$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{(-1)^{1+1}\begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{D} & \frac{(-1)^{1+2}\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{D} & \frac{(-1)^{1+3}\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{D} \\ \frac{(-1)^{2+1}\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{D} & \frac{(-1)^{2+2}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{D} & \frac{(-1)^{2+3}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{D} \\ \frac{(-1)^{3+1}\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \end{vmatrix}}{D} & \frac{(-1)^{3+2}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}}{D} & \frac{(-1)^{3+3}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}}{D} \end{bmatrix}^{T}$$

$$= \begin{bmatrix} \frac{\begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{r^2 \sin\theta} \\ -\frac{\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{r^2 \sin\theta} \\ \frac{\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}}{r^2 \sin\theta} \end{bmatrix}^{T} = \begin{bmatrix} \frac{\begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} \\ -\frac{\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}}{r^2 \sin\theta} \\ \frac{\begin{vmatrix} \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}}{r^2 \sin\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\begin{vmatrix} r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ -r\sin\theta & 0 \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ -r\sin\theta & 0 \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}}{r^2 \sin\theta} \\ -\frac{\begin{vmatrix} \sin\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & 0 \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \cos\theta & 0 \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}}{r^2 \sin\theta} \\ \frac{\begin{vmatrix} \sin\theta\sin\varphi & r\cos\theta\sin\varphi \\ \cos\theta & -r\sin\theta \end{vmatrix}}{r^2 \sin\theta} & -\frac{\begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi \\ \cos\theta & -r\sin\theta \end{vmatrix}}{r^2 \sin\theta} & \frac{\begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi \end{vmatrix}}{r^2 \sin\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{\cos\theta\cos\varphi}{r} & \frac{\cos\theta\sin\varphi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & \frac{0}{r\sin\theta} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{\cos\theta\cos\varphi}{r} & \frac{\cos\theta\sin\varphi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{bmatrix}$$

Jadi:
$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix}^{-1} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{\cos\theta\cos\varphi}{r} & \frac{\cos\theta\sin\varphi}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{bmatrix}$$

Dengan sifat keidentikan akan didapatkan harga sebagai berikut:

$$\frac{\partial r}{\partial x} = \sin\theta\cos\varphi \qquad \frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\varphi}{r} \qquad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}$$
$$\frac{\partial r}{\partial y} = \sin\theta\sin\varphi \rightarrow \frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\varphi}{r} \rightarrow \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}$$
$$\frac{\partial r}{\partial z} = \cos\theta \qquad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r} \qquad \frac{\partial \varphi}{\partial z} = 0$$

Hubungan koordinat kebalikkan kartesian dan koordinat kebalikkan bola:

Hubungan yang dimaksud adalah: $\frac{1}{\partial x} \xrightarrow{B} \frac{1}{\partial r}, \frac{1}{\partial y} \xrightarrow{B} \frac{1}{\partial \theta}, \frac{1}{\partial z} \xrightarrow{B} \frac{1}{\partial \varphi}$ yang diperoleh dari persamaan berikut:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi = dr\frac{\partial x}{\partial r} + d\theta\frac{\partial x}{\partial \theta} + d\varphi\frac{\partial x}{\partial \varphi} = \left(dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial \theta} + d\varphi\frac{\partial}{\partial \varphi}\right)x$$

$dx = \left(dr\frac{\partial}{\partial r} + d\theta\frac{\partial}{\partial \theta} + d\varphi\frac{\partial}{\partial \varphi}\right)x \rightarrow \partial x = \left(\partial r\frac{\partial}{\partial r} + \partial\theta\frac{\partial}{\partial \theta} + \partial\varphi\frac{\partial}{\partial \varphi}\right)x$, sehingga:

$$\frac{\partial}{\partial x}x = \frac{1}{\partial x}\partial x = \frac{1}{\partial x}\left(\partial r\frac{\partial}{\partial r} + \partial\theta\frac{\partial}{\partial \theta} + \partial\varphi\frac{\partial}{\partial \varphi}\right)x = \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\right)x$$

Atau: $\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}$, dengan cara yang sama akan didapatkan bahwa:
$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}$ dan $\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}$
Penggabungan persamaan ini adalah:

$$\left.\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial y}&=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial z}&=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}\end{aligned}\right\}\text{ sehingga: }\begin{bmatrix}\frac{1}{\partial x}\\ \frac{1}{\partial y}\\ \frac{1}{\partial z}\end{bmatrix}=\begin{bmatrix}\frac{\partial r}{\partial x}&\frac{\partial\theta}{\partial x}&\frac{\partial\varphi}{\partial x}\\ \frac{\partial r}{\partial y}&\frac{\partial\theta}{\partial y}&\frac{\partial\varphi}{\partial y}\\ \frac{\partial r}{\partial z}&\frac{\partial\theta}{\partial z}&\frac{\partial\varphi}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\varphi}\end{bmatrix}\text{ dimana: }\begin{bmatrix}\frac{\partial r}{\partial x}&\frac{\partial\theta}{\partial x}&\frac{\partial\varphi}{\partial x}\\ \frac{\partial r}{\partial y}&\frac{\partial\theta}{\partial y}&\frac{\partial\varphi}{\partial y}\\ \frac{\partial r}{\partial z}&\frac{\partial\theta}{\partial z}&\frac{\partial\varphi}{\partial z}\end{bmatrix}$$

$$=\begin{bmatrix}\sin\theta\cos\varphi&\frac{\cos\theta\cos\varphi}{r}&-\frac{\sin\varphi}{r\sin\theta}\\ \sin\theta\sin\varphi&\frac{\cos\theta\sin\varphi}{r}&\frac{\cos\varphi}{r\sin\theta}\\ \cos\theta&-\frac{\sin\theta}{r}&0\end{bmatrix}\text{ karena: }\begin{bmatrix}\frac{1}{\partial x}\\ \frac{1}{\partial y}\\ \frac{1}{\partial z}\end{bmatrix}=\begin{bmatrix}\frac{\partial r}{\partial x}&\frac{\partial\theta}{\partial x}&\frac{\partial\varphi}{\partial x}\\ \frac{\partial r}{\partial y}&\frac{\partial\theta}{\partial y}&\frac{\partial\varphi}{\partial y}\\ \frac{\partial r}{\partial z}&\frac{\partial\theta}{\partial z}&\frac{\partial\varphi}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\varphi}\end{bmatrix}$$

$$=\begin{bmatrix}\sin\theta\cos\varphi&\frac{\cos\theta\cos\varphi}{r}&-\frac{\sin\varphi}{r\sin\theta}\\ \sin\theta\sin\varphi&\frac{\cos\theta\sin\varphi}{r}&\frac{\cos\varphi}{r\sin\theta}\\ \cos\theta&-\frac{\sin\theta}{r}&0\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial\varphi}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\\ \sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\\ \cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\end{bmatrix}$$

$$\text{atau: }\begin{bmatrix}\frac{1}{\partial x}\\ \frac{1}{\partial y}\\ \frac{1}{\partial z}\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\\ \sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\\ \cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\end{bmatrix}$$

$$\text{Sehingga: }\left.\begin{aligned}\frac{\partial}{\partial x}&=\sin\theta\cos\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial y}&=\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial z}&=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(3)$$

Komutator Dalam Koordinat Bola:
Berdasarkan persamaan persamaan (3) ini, maka dapat dibuktikan bahwa:

1. Buktikanlah bahwa:
   $\hat{L}_x=-i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_y=-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$ dan $\hat{L}_z=-i\hbar\frac{\partial}{\partial\varphi}$
   Bukti:
   $\hat{L}_x=-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)=-i\hbar\left\{y\left(\frac{\partial}{\partial z}\right)-z\left(\frac{\partial}{\partial y}\right)\right\}=-i\hbar\left\{r\sin\theta\sin\varphi\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\right.$
   $\left.-r\cos\theta\left(\sin\theta\sin\varphi\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right\}=-i\hbar\left\{r\sin\theta\sin\varphi\cos\theta\frac{\partial}{\partial r}\right.$
   $\left.-r\sin\theta\sin\varphi\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}-r\cos\theta\sin\theta\sin\varphi\frac{\partial}{\partial r}-r\cos\theta\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}-r\cos\theta\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right\}$
   $=-i\hbar\{r(\sin\theta\cos\theta\sin\varphi-\sin\theta\cos\theta\sin\varphi)\frac{\partial}{\partial r}-(\sin\theta\sin\theta+\cos\theta\cos\theta)\sin\varphi\frac{\partial}{\partial\theta}$
   $\left.-\frac{\cos\theta}{\sin\theta}\cos\varphi\frac{\partial}{\partial\varphi}\right\}=-i\hbar\left\{r(0)\frac{\partial}{\partial r}-(\sin^2\theta+\cos^2\theta)\sin\varphi\frac{\partial}{\partial\theta}-\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right\}$
   $=-i\hbar\left\{-\sin\varphi\frac{\partial}{\partial\theta}-\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right\}$
   Jadi: $\hat{L}_x=-i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\rightarrow$ terbukti.
   Dengan cara yang sama dapat dibuktikan bahwa: $\hat{L}_y=-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$ dan $\hat{L}_z=-i\hbar\frac{\partial}{\partial\varphi}$
2. Buktikanlah bahwa: $\hat{L}^2=-\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$

Bukti:

$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z$ = A + B + C, dimana:

A = $\hat{L}_x\hat{L}_x$, B = $\hat{L}_y\hat{L}_y$ dan C = $\hat{L}_z\hat{L}_z$

A = $\hat{L}_x\hat{L}_x = \left\{-i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}\left\{-i\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right\}$

$= (-i\hbar)^2\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$= \left(-\hbar^2\sin\varphi\frac{\partial}{\partial\theta} - \hbar^2\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$= -\hbar^2\sin\varphi\frac{\partial}{\partial\theta}\left(\sin\varphi\frac{\partial}{\partial\theta}\right) - \hbar^2\sin\varphi\frac{\partial}{\partial\theta}\left(\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) - \hbar^2\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right)$

$-\hbar^2\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) = -\hbar^2\sin\varphi\sin\varphi\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right) - \hbar^2\sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\varphi}\right)$

$-\hbar^2\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right) - \hbar^2\cot\theta\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) = -\hbar^2\sin^2\varphi\frac{\partial^2}{\partial\theta^2}$

$-\hbar^2\sin\varphi\cos\varphi\left\{-\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}\right\} - \hbar^2\cot\theta\cos\varphi\left\{\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right\}$

$-\hbar^2\cot^2\theta\cos\varphi\left\{-\sin\varphi\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial^2}{\partial\varphi^2}\right\} = -\hbar^2\sin^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\hbar^2\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}$

$-\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi} - \hbar^2\cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} - \hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta} +$

$\hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}$

$-\hbar^2\cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2} = -\hbar^2\sin^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\hbar^2\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} -$

$2\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}$

$-\hbar^2\cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}$

B = $\hat{L}_y\hat{L}_y = \left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}\left\{-i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\right\}$

$= (-i\hbar)^2\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$

$= -\hbar^2\cos^2\varphi\frac{\partial^2}{\partial\theta^2} - \hbar^2\frac{1}{\sin^2\theta}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi} - \hbar^2\cot\theta\sin^2\varphi\frac{\partial}{\partial\theta}$

$+ \hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta} - \hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}$

$= -\hbar^2\cos^2\varphi\frac{\partial^2}{\partial\theta^2} - \hbar^2\frac{1}{\sin^2\theta}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + 2\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi} - \hbar^2\cot\theta\sin^2\varphi\frac{\partial}{\partial\theta}$

$-\hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}$

A + B $= -\hbar^2\sin^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\hbar^2\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - 2\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}$

$-\hbar^2\cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}$

$-\hbar^2\cos^2\varphi\frac{\partial^2}{\partial\theta^2} - \hbar^2\frac{1}{\sin^2\theta}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + 2\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi}$

$-\hbar^2\cot\theta\sin^2\varphi\frac{\partial}{\partial\theta} - \hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}$

$= \left(-\hbar^2\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \hbar^2\cos^2\varphi\frac{\partial^2}{\partial\theta^2}\right) + \left(\frac{1}{\sin^2\theta}\hbar^2\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\frac{1}{\sin^2\theta}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$+ \left(2\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi} - 2\hbar^2\cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right) - \left(\hbar^2\cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} +\right.$

$\left.\hbar^2\cot\theta\sin^2\varphi\frac{\partial}{\partial\theta}\right) + \left(\hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$-\left(\hbar^2\cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2} + \hbar^2\cot^2\theta\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)$

$= -\hbar^2(\sin^2\varphi + \cos^2\varphi)\frac{\partial^2}{\partial\theta^2} + 0 + 0 - \hbar^2\cot\theta(\cos^2\varphi + \sin^2\varphi)\frac{\partial}{\partial\theta} + 0$

$-\hbar^2\cot^2\theta(\cos^2\varphi + \sin^2\varphi)\frac{\partial^2}{\partial\varphi^2} = -\hbar^2\frac{\partial^2}{\partial\theta^2} - \hbar^2\cot\theta\frac{\partial}{\partial\theta} - \hbar^2\cot^2\theta\frac{\partial^2}{\partial\varphi^2}$

$= -\hbar^2\cot\theta\frac{\partial}{\partial\theta} - \hbar^2\frac{\partial^2}{\partial\theta^2} - \hbar^2\cot^2\theta\frac{\partial^2}{\partial\varphi^2} = -\hbar^2\left(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \cot^2\theta\frac{\partial^2}{\partial\varphi^2}\right)$

$= -\hbar^2 \left(\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\sin\theta}{\sin\theta}\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\,\frac{\partial^2}{\partial\varphi^2}\right) = -\hbar^2 \left[\frac{1}{\sin\theta}\left(\cos\theta\,\frac{\partial}{\partial\theta} + \sin\theta\,\frac{\partial^2}{\partial\theta^2}\right) + \cot^2\theta\,\frac{\partial^2}{\partial\varphi^2}\right]$

$= -\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \cot^2\theta\,\frac{\partial^2}{\partial\varphi^2}\right\}$

Atau: A + B = $-\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \cot^2\theta\,\frac{\partial^2}{\partial\varphi^2}\right\}$

C = $\hat{L}_z\hat{L}_z = \left(-i\hbar\frac{\partial}{\partial\varphi}\right)\left(-i\hbar\frac{\partial}{\partial\varphi}\right) = (-i\hbar)^2\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right) = -\hbar^2\,\frac{\partial^2}{\partial\varphi^2}$

Sehingga:

$\hat{L}^2$ = A + B + C = $-\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \cot^2\theta\,\frac{\partial^2}{\partial\varphi^2}\right\} - \hbar^2\,\frac{\partial^2}{\partial\varphi^2}$

$= -\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \cot^2\theta\,\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\varphi^2}\right\} = -\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + (\cot^2\theta + 1)\,\frac{\partial^2}{\partial\varphi^2}\right\}$

$= -\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \left(\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2}\right\} = \hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2}\right\}$

$= -\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$

Jadi:

$\hat{L}^2 = -\hbar^2 \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$ → terbukti

3. Buktikanlah bahwa: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$ dan $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$

Bukti:

$[\hat{L}_x, \hat{L}_y]\psi = [\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x]\psi = \hat{L}_x\hat{L}_y\psi - \hat{L}_y\hat{L}_x\psi$

$= -\hbar^2\left(\sin\varphi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi}\right)\left(-\cos\varphi\,\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi}\right)\psi$

$+ \hbar^2\left(\cos\varphi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi}\right)\left(-\sin\varphi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi}\right)\psi$

$= \hbar^2\cos\varphi\sin\varphi\,\frac{\partial^2}{\partial\theta^2}\psi - \hbar^2\sin\varphi\sin\varphi\,\frac{\partial}{\partial\theta}\left(\cot\theta\,\frac{\partial}{\partial\varphi}\right)\psi + \hbar^2\cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi}\left(\cos\varphi\,\frac{\partial}{\partial\theta}\right)\psi$

$-\hbar^2\cot\theta\cot\theta\cos\varphi\,\frac{\partial}{\partial\varphi}\left(\sin\varphi\,\frac{\partial}{\partial\varphi}\right)\psi - \hbar^2\sin\varphi\cos\varphi\,\frac{\partial^2}{\partial\theta^2}\psi - \hbar^2\cos\varphi\cos\varphi\,\frac{\partial}{\partial\theta}\left(\cot\theta\,\frac{\partial}{\partial\varphi}\right)\psi$

$+ \hbar^2\cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi}\left(\sin\varphi\,\frac{\partial}{\partial\theta}\right)\psi + \hbar^2\cot\theta\cot\theta\sin\varphi\,\frac{\partial}{\partial\varphi}\left(\cos\varphi\,\frac{\partial}{\partial\varphi}\right)\psi$

$= \left(\hbar^2\cos\varphi\sin\varphi\,\frac{\partial^2}{\partial\theta^2} - \hbar^2\sin\varphi\cos\varphi\,\frac{\partial^2}{\partial\theta^2}\right)\psi - \hbar^2(\cos^2\varphi + \sin^2\varphi)\,\frac{\partial}{\partial\theta}\left(\cot\theta\,\frac{\partial}{\partial\varphi}\right)\psi$

$+ \hbar^2\cot^2\theta\sin\varphi\left\{\frac{\partial}{\partial\varphi}(\cos\varphi)\,\frac{\partial}{\partial\varphi} + \cos\varphi\,\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right)\right\}\psi - \hbar^2\cot^2\theta\cos\varphi\left\{\frac{\partial}{\partial\varphi}(\sin\varphi)\,\frac{\partial}{\partial\varphi} +\right.$

$\left.\sin\varphi\,\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right)\right\}\psi + \hbar^2\cot\theta\sin\varphi\left\{\frac{\partial}{\partial\varphi}(\sin\varphi)\,\frac{\partial}{\partial\theta} + \sin\varphi\,\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta}\right)\right\}\psi$

$+ \hbar^2\cot\theta\cos\varphi\left\{\frac{\partial}{\partial\varphi}(\cos\varphi)\,\frac{\partial}{\partial\theta} + \cos\varphi\,\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta}\right)\right\}\psi = 0\psi + \hbar^2\,\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\psi$

$-\hbar^2\cot\theta\,\frac{\partial^2}{\partial\theta\partial\varphi}\psi - \hbar^2\cot^2\theta\,\frac{\partial}{\partial\varphi}\psi + 0\psi + 0\psi + \hbar^2\cot\theta\,\frac{\partial^2}{\partial\varphi\partial\theta}\psi$

$= 0 + \hbar^2\,\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\psi + \left(\hbar^2\cot\theta\,\frac{\partial^2}{\partial\theta\partial\varphi}\psi - \hbar^2\cot\theta\,\frac{\partial^2}{\partial\theta\partial\varphi}\psi\right) - \hbar^2\cot^2\theta\,\frac{\partial}{\partial\varphi}\psi + 0 + 0$

$+ \hbar^2\cot\theta\,\frac{\partial^2}{\partial\varphi\partial\theta}\psi = \hbar^2\,\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\psi + 0 - \hbar^2\cot^2\theta\,\frac{\partial}{\partial\varphi}\psi = \hbar^2\,\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \hbar^2\cot^2\theta\,\frac{\partial}{\partial\varphi}\psi =$

$\hbar^2\,\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\psi - \hbar^2\,\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}\psi = \hbar^2\left\{\frac{1}{\sin^2\theta}(1 - \cos^2\theta)\right\}\frac{\partial}{\partial\varphi}\psi = \hbar^2\left(\frac{\sin^2\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi}\psi$

$= \hbar^2\,\frac{\partial}{\partial\varphi}\psi = -(-\hbar^2)\,\frac{\partial}{\partial\varphi}\psi = -(i^2\hbar^2)\,\frac{\partial}{\partial\varphi}\psi = -(i\hbar)^2\,\frac{\partial}{\partial\varphi}\psi = -(-i\hbar)^2\,\frac{\partial}{\partial\varphi}\psi$

$= -(-i\hbar)(-i\hbar)\,\frac{\partial}{\partial\varphi}\psi = i\hbar\left(-i\hbar\,\frac{\partial}{\partial\varphi}\right)\psi = i\hbar\hat{L}_z\psi$

Jadi: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ → terbukti.

Dengan cara yang sama dapat dibuktikan bahwa: $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$ dan $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$

Jadi: $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$ dan $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$

PENUTUP

Simpulan

Disini telah diuraikan dalam bentuk soal pembuktian tentang operator komutator dalam 2 sistim koordinat, yaitu:

Koordinat kartesian:

1. $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = [\hat{x}, \hat{p}] = [\hat{y}, \hat{p}] = [\hat{z}, \hat{p}] = i\hbar$
2. $[\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_x] = [\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = [\hat{L}^2, \hat{L}] = 0$
3. $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$, $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$, $\hat{L} \times \hat{L} = i\hbar\hat{L}$ dan $\vec{L} \times \vec{L} = \vec{0}$

Koordinat bola:

1. $\hat{L}_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$, $\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$
2. $\hat{L}^2 = -\hbar^2\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$

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